Mean-field Monte Carlo approach to the Sherrington-Kirkpatrick model with asymmetric couplings

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An alternative method is applied to the nonequilibrium zero temperature dynamics of a Sherrington-Kirkpatrick model with nonsymmetric random exchange couplings $J_{ij} \neq J_{ji}$. Based on exact stochastic single spin equations for the infinite system, this mean-field Monte Carlo approach avoids the strong finite size effects of the usual simulations for this problem. Varying the average symmetry $\eta = [J_{ij}J_{ji}]/[J_{ij}^2]$, we find a clear transition from ergodic dynamics at $\eta < \eta_c = 0.825$ to a phase $(\eta > \eta_c)$ where a finite fraction of the spins freezes. Only in the fully symmetric case $\eta = 1$ is the entire system frozen.

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I. INTRODUCTION

Models composed of spinlike elements interacting via long range competing interactions play an important role in the modeling of complex systems such as spin glasses [1] and neural networks [2].

The dynamics of such models is usually described by a process where each spin S_i at site i is flipped into the direction of its internal field $h_i = \sum_{j \neq i} J_{ij} S_j$ with a probability depending on $S_i h_i$. For the spin-glass applications the strengths J_{ij} of the couplings are symmetric, i.e., $J_{ij} = J_{ji}$. A Glauber dynamics leads to a Gibbs distribution of the spins with a Hamiltonian $H = -\frac{1}{2}\sum_{i,j} J_{ij} S_i S_j$. The temporal behavior of the model at zero temperature can be understood as a relaxation process into one of the many valleys of a complex energy landscape which are separated by large barriers.

In the context of neural network models, spins are usually interpreted as two-state neurons and the couplings J_{ij} as the synaptic efficacies [2]. In this case the physical assumption of symmetric couplings is not natural and has often been questioned as being too restrictive for biological models.

Giving up the restriction of symmetry and the existence of a Hamiltonian opens up the possibility of more complicated dynamical behavior in the noise-free (zero temperature) case. A deterministic dynamics on a phase space of finitely many (2^N) states can only end in *periodic attractors*. Nevertheless, the temporal development of asymmetric models at large N can exhibit many features of a stochastic dynamics. This happens when the period length of the attractor and the transient time to reach the attractor diverge with the system size. In such cases, the system can explore large regions in phase space. Thus it was argued that asymmetry might play a similar role as temperature, possibly leading to a transition from a frozen state to an ergodic behavior.

To understand the generic physics of asymmetric spin models, a modification of the well known Sherrington-Kirkpatrick [1] model of spin glasses was studied by many authors. In this model the degree of symmetry can be varied continuously by a parameter η. $\eta = 1$ means fully symmetric couplings and $\eta = 0$ corresponds to the case where J_{ij} and J_{ji} are totally uncorrelated.

Unfortunately, the absence of a Hamiltonian and related basic principles of thermal equilibrium such as detailed balance make applications of standard analytical tools of disorder physics, such as the replica method, impossible. Exact analytical solutions to systems with asymmetry are restricted to simpler cases. These include spins in a one-dimensional chain or on a Cayley tree [3,4], a spherical model [5], and the fully uncorrelated case $\eta = 0$ [6–8]. A perturbative treatment of asymmetry has been discussed in [9,10]. Also, exact results for a "soft-spin" model of game theory involving random asymmetric interactions could be obtained [11].

However, most results on the general asymmetric SK model are based on simulations and can be summarized as follows: An investigation of the model at *nonzero* temperature by Crisanti and Sompolinsky predicts the absence of a spin-glass phase for any amount of asymmetry $(\eta < 1)$ [6], a result which is in accordance with the corresponding spherical model and a perturbative treatment by Hertz, Grinstein, and Solla [9]. On the other hand, the situation at zero temperature seems less clear. Extensive numerical simulations yield different types of dynamical behavior depending on the scaling of observation times with the size of the system.

In a first type of numerical experiment, one waits long enough until the dynamics reaches a periodic attractor. For $\eta = 1$, only fixed points, or, in the case of a synchronous update of spins, cycles of length 2 can appear. As first shown by Gutfreund, Reger, and Young [12], the introduction of a small asymmetry does not change this picture much. Fixed points and short cycles still dominate the dynamics. Recent simulations by Nützel and Krey [13] performed with large N and parallel updates, reveal that *all* initial conditions flow into cycles of length two when η is greater than about 0.5. Below $\eta \approx 0.5$, the typical period length of the attractors was found to increase exponentially with the number of spins [14]. However, this transition can be observed only after an exponentially long transient time which is needed to reach the attractors [13]. On time scales which do not diverge with N, a sufficiently large system will be found in a *transient state.* We will be concerned only with this time scale for the rest of the paper.

In a second type of numerical experiment, one tries to simulate the system during its transient, corresponding to a situation, where the limit $N\to\infty$ is performed first, before the long time limit. For small η, after a long enough waiting time t_0, Crisanti and Sompolinsky [6] find that the system relaxes into a stationary state, where the correlations $(1/N)\sum_{i=1}^{N}S_i(t+t_0)S_i(t_0)$ of a spin S_i at two times t_0 and t_0+t depend only on the time lag t. These correlations decay to zero with a finite correlation time τ. The complexity of a spin trajectory $S_i(t)$, measured by its Shannon entropy h, was found to decrease [15] with growing η. However, by the drastic increase of t_0 for larger η, reliable estimates of h are not available. Extrapolations indicate that the entropy might vanish, and the system freezes into an ordered state for $\eta\approx 0.6$.

A third type of simulation avoids the waiting time t_0 by studying the nonequilibrium relaxation of the spins right from $t=0$. Using large scale simulations for the model with sequential dynamics, Spitzner and Kinzel [16] observe an additional effect, a "freezing transition" for the remanent magnetization. A nonzero remanent magnetization, which is an indicator of a nonergodic dynamics, was found for $\eta>0.83$. Above this value the system seems to freeze. For parallel updates, a similar value for the critical symmetry ($\eta=0.85$) was obtained by Pfenning, Rieger, and Schreckenberg [17].

Unfortunately, simulations of the model show strong finite size effects [18], which make an extrapolation to the thermodynamic limit $N\to\infty$ a highly nontrivial task. Since one has to store about N^2 couplings for a fully connected model, the simulation of large systems becomes increasingly tedious.

On the other hand, fully connected models have the feature that an exact dynamical mean-field theory can be formulated for the *infinite system.* Recently, [19] this fact has been used to develop an alternative method for simulating the parallel dynamics of disordered systems with long range interactions avoiding finite size problems. In a previous Letter, the method was applied to the remanent magnetization of the symmetric ($\eta=1$) SK model. Using generating functions, we derived stochastic single spin equations which yield an *exact description* of the system's dynamics for $N=\infty$. To calculate disorder averaged quantities, the single spin dynamics was simulated using a Monte Carlo procedure. In this paper we apply our method to the asymmetric SK model.

The paper is organized as follows. In Sec. II we introduce the model and its basic properties. A generating function is used in Sec. III to construct a mean-field theory for the dynamics. In Sec. IV the Monte Carlo method for the single-site stochastic dynamics is described. The results of our simulations are presented and discussed in Sec. V. We end the paper with a summary and an outlook in Sec. VI.

II. THE SK MODEL WITH ASYMMETRIC INTERACTIONS

The model consists of N Ising spins $S_i=\pm 1$, where every spin S_i is connected to all other spins S_j by couplings J_{ij}, which are independent Gaussian random variables for all $i<j$ with distribution $P(J_{ij})=\sqrt{N/2\pi}\exp[-(N/2)J_{ij}^2]$. Additionally the symmetry of the matrix of couplings is given by the average symmetry parameter n:

$$[J_{ij}J_{ji}]=\eta/N \ , \tag{1}$$

where the brackets denote an average over the distribution of couplings. This means that the couplings are fully antisymmetric ($J_{ij}=-J_{ji}$) if $\eta=-1$ and totally uncorrelated if $\eta=0$. Finally, symmetric couplings ($J_{ij}=J_{ji}$), which correspond to the spin-glass model, are given by $\eta=1$.

Couplings with these symmetry properties can be constructed via

$$J_{ij}=\left[\frac{1+\eta}{2}\right]^{1/2}J_{ij}^{s}+\left[\frac{1-\eta}{2}\right]^{1/2}J_{ij}^{a} \ , \tag{2}$$

with

$$J_{ij}^{s}=J_{ji}^{s} \quad \text{and} \quad J_{ij}^{a}=-J_{ji}^{a} \ .$$

We shall restrict ourselves in this paper to the simplest type of noise-free dynamics for the model, which consists of a *synchronous* update of all spins at an elementary time step t:

$$S_i(t+1)=\text{sgn}[h_i(t)] \ , \quad i=1,\ldots,N \ , \tag{3}$$

where the internal field of the spin S_i is given by

$$h_i(t)=\sum_{j\neq i}J_{ij}S_j(t) \ . \tag{4}$$

As for the sequential type of dynamics it can be shown [20] that for $\eta=1$ the parallel updating has a nonincreasing Lyapunov function $\mathcal{L}$:

$$\mathcal{L}=-\sum_i \text{sgn}\,(h_i(t))h_i(t) \ . \tag{5}$$

For this case the system approaches a state at large times t for which $S_i(t)=S_i(t+2)$ with probability 1. This includes fixed points as well as cycles of length 2. The case $\eta=1$ also allows for the definition of a noisy dynamics that obeys detailed balance. Then a Gibbs distribution can be constructed [21], and equilibrium properties can be calculated with the help of the replica trick. Such a calculation has been performed by Fontanari and Köberle [22] for the Little model, which is the synchronous version of the Hopfield model. Unfortunately, for all $\eta\neq 1$, Lyapunov functions or equilibrium distributions are not available.

Another interesting limiting case of the model is found for $\eta=0$. For $N\to\infty$, this case can be treated analytically [6,8,7], showing a rather stochastic type of dynamics. For synchronous updates, correlations decay to zero just after one time step. Although the system is a deterministic finite state automaton, a finite trajectory of spins in the infinite system at a given site i appears as a completely random sequence of values ± 1, like the tosses of an ideal coin.

In this paper, we will not try to develop a self-consistent dynamical theory for the steady-state behavior at infinite times. This was even a highly nontrivial task for models with symmetric couplings, [23] and required the introduction of a sequence of time scales all diverging with N. Instead, we will be concerned with the transient behavior of the model. To be specific, we shall study the *initial value problem*, where at time $t=0$ all spins $S_i(0)$ are uncorrelated to the couplings J_{ij}. For a spin glass such a state could be prepared by heating the system up to infinite temperatures, destroying all correlations to previous times and then cooling it immediately to zero temperature. Alternatively we could switch off an infinitely strong magnetic field.

III. DYNAMICAL MEAN-FIELD THEORY

In the following we will use the fact that each spin $S_i(t)$ is coupled to *all other* spins. It is reasonable to assume that the internal field $h_i(t)$ at the location of spin i can be replaced by an effective "mean field," which does not depend explicitly on the state of the other $N-1$ spins $S_j(t)$ when the system size N goes to infinity.

This is in fact the case. However, this effective field for disordered models like the present one appears to be a rather complex time-dependent *random process*, being *different from the averaged* internal field $[h_i(t)]$.

In this section we shall construct random process for the effective field $h_i(t)$ explicitly, which will be used later to generate stochastic spin trajectories in a Monte Carlo method.

We use the technique of *dynamical generating functions*, which are a type of path integral that are very convenient for the description of stochastic dynamical systems [24–26].

Assume that we are interested in the statistical properties of a finite, possibly large number N_T of spin trajectories of length t_f, at the sites $i=1,\ldots,N_T$, in a system where the total number N of spins diverges.

These properties can be derived from the *generating function* $[Z(\mathbf{l})]$ for the internal fields $h_i(t)$, $i=1,\ldots,N_T$, $t=0,\ldots,t_f$.

$$[Z(\mathbf{l})]=\left[\mathrm{Tr}_{S(t)}\int\prod_{i=1}^{N}\prod_{t=0}^{t_f}\left\{dh_i(t)\Theta(S_i(t+1)h_i(t))\delta\left[h_i(t)-\sum_{j\neq i}\left[\left[\frac{1+\eta}{2}\right]^{1/2}J_{ij}^{s}+\left[\frac{1-\eta}{2}\right]^{1/2}J_{ij}^{a}\right]S_j(t)\right]\right\}\times\exp\left[i\sum_{t=0}^{t_f}\sum_{i=1}^{N_T}l_i(t)h_i(t)\right]\right]_J . \tag{6}$$

$[\ ]_J$ denotes an average over the random couplings, and Tr_S is the sum over all 2^{Nt_f} possible combinations of the spin states $S_i(t)=\pm 1$. $\theta(x)$ is the unit step function. By construction, only those "spin paths" $S_i(t)$ contribute to $Z(\mathbf{l})$ which fulfill the correct equations of motion (3) and (4).

Using the vector $\mathbf{l}$ of parameters $l_i(t)$ conjugate to the internal fields $h_i(t)$, the complete joint probability density of the $h_i(t)$ can be reconstructed via Fourier transforms.

The calculation of $[Z(\mathbf{l})]_J$ in the limit $N\to\infty$ closely follows the derivation given by Henkel and Opper [27] for the synchronous dynamics of a neural network, which in turn is based on the treatment of Sompolinsky and Zippelius [24] for the dynamic mean-field theory of spin glasses. The formal steps of the calculation are given in Appendix A.

We find that in the thermodynamic limit the generating function completely factorizes into independent components for the N_T spins:

$$[Z(\mathbf{l})]_J\propto\prod_{i=1}^{N_T}\left\langle\mathrm{Tr}_{S_i(t)}\int\prod_t\{dh_i(t)\Theta(S_i(t+1)h_i(t))\}\exp\left[i\sum_t(l_i(t)h_i(t))\right]\prod_t\delta\left[h_i(t)-\phi_i(t)-\eta\sum_s K(t,s)S_i(s)\right]\right\rangle_{\phi_i} . \tag{7}$$

From this representation, we see that the spin dynamics (3) is described by the uncorrelated system of stochastic dynamical equations:

$$S_i(t+1)=\mathrm{sgn}(h_i(t)) , \tag{8}$$

with

$$h_i(t)=\phi_i(t)+\eta\sum_{s<t}K(t,s)S_i(s) .$$

In the mean-field picture the time-independent random couplings to other spins are replaced by Gaussian noise terms $\phi_i(t)$ with zero means and correlations

$$\langle\phi_i(t)\phi_j(s)\rangle_\phi=\delta_{ij}\langle S_i(t)S_i(s)\rangle_\phi=\delta_{ij}C(t,s) . \tag{9}$$

In addition, deterministic interactions $K(t,s)$ with spins $S_i(s)$ at the *same site* but for all previous times $s<t$ appear:

$$K(t,s)=\left\langle\frac{\partial}{\partial\phi(s)}S(t)\right\rangle_\phi \tag{10}$$

Replacing all spin variables recursively by spins and Gaussian variables at earlier times, we find that for each time t the value of the spin $S_i(t)$ is a function of the random variables $S_i(0)$ and the $t-1$ Gaussian fields $\phi_i(t)$. Independent realizations of the stochastic spin trajectories mimic the spin flips at a finite number of sites in a "real" single infinite SK model.

The stochastic process (8) is completely determined if the sets of order parameters $K(t,s)$ and $C(t,s)$ are known. These in turn are determined from averages over fields $\phi(s)$ for times $s<t$ only.

We will end this section with exact relations for Gaussian averages which can be obtained directly from the stochastic dynamics.

The first relation helps us to express the order parameter $K(t,s)$ by the correlation function $\langle S(t)\phi(s)\rangle$, which is easier to estimate than the average of the partial derivative (10). It is based on a simple theorem for Gaussian random variables. (It is a discrete version of *Novikov's theorem* [28], well known in the theory of stochastic processes). Using Appendix B, we show that

$$\langle S(t)\phi(s)\rangle = \sum_{\tau=0}^{t} K(t,\tau)C(\tau,s) \ , \tag{11}$$

which holds independently of the parameter η. On the other hand, a fluctuation dissipation theorem, which would enable us to express $K(t,s)$ directly by $C(t,s)$, is not available in the asymmetric models.

Relation (11) can be used to show (see Appendix C) that the correlation function $C(t,s)=0$, whenever $|t-s|$ is odd. This can be expressed symbolically as $C(\text{odd, even})=C(\text{even, odd})=0$. As a consequence, the magnetization $m(t)=\langle S(t)S(0)\rangle$ at odd time steps vanishes (see also [29]).

Since a Gaussian process is completely determined by its first two moments, we conclude that the sets of variables ϕ(odd) and ϕ(even) are independent. Second, we prove in Appendix C that $K(\text{odd, odd}')=K(\text{even , even}')=0$. Inserting these relations into (8) shows that the spins S(even) only depend on spins $S(s)$ with $s=$even, and fields $\phi(s')$ with $s'=$odd. Conversely S(odd) is a function of the Gaussian fields ϕ(even) alone. Thus spins for odd- and even-time steps are completely decoupled and *independent* random variables.

IV. MONTE CARLO SIMULATION

The single spin equations (8) can be used to calculate *exact* averages for $N=\infty$ by expressing the spin variables as explicit functions of the Gaussian fields $\phi(t)$ and performing integrations weighted by the multivariate Gaussian measure. This integration is most conveniently performed by a Monte Carlo process, where a sequence of Gaussian random numbers with respect to the covariance $C(t,s)$ is generated and a trajectory of spins $S(t)$ is created via Eq. (8). The necessary averages in each time step can be estimated by summing over a large number N_T of trajectories. N_T should *not be confused with N, the number of spins in the model,* which equals infinity.

The algorithm for the Monte Carlo simulation can be described as follows. (For convenience we write the time indices as subscripts in the matrix below.) Sample averages will be denoted by overbars.

1. Start at $t=0$:

(a) Set $S^k(0)=1$ for all $k=1,\dots,N_T$, where N_T denotes the number of spin trajectories.

(b) Set $h^k(0)=\phi^k(0)$, where $\phi^k(0)$'s are drawn independently from the distribution $P(\phi^k(0))=(1/\sqrt{2\pi})\exp[-(\phi^k(0))^2/2]$.

2. An arbitrary time step t:

(a) Evaluate the spins at time t from the dynamical equations:

$$S^k(t)=\text{sgn}(h^k(t-1)) \quad \text{for } k=1,\dots,N_T \ . \tag{12}$$

Note that the internal field $h^k(t-1)$ was already calculated at time step $t-1$.

(b) Calculate the sample averages over all trajectories:

$$\overline{S_t S_\tau} \quad \text{for } \tau=0,\dots,t-1 \ , \tag{13}$$

which give the correlation matrix for the Gaussian noise variables:

$$\hat{\Phi}_t = \begin{bmatrix} \overline{\phi_0\phi_0} & \overline{\phi_1\phi_0} & \cdots & \overline{\phi_t\phi_0} \\ \overline{\phi_1\phi_0} & \overline{\phi_1\phi_1} & & \\ \vdots & & \ddots & \vdots \\ \overline{\phi_t\phi_0} & \cdots & & \overline{\phi_t\phi_t} \end{bmatrix} = \begin{bmatrix} 1 & \overline{S_1S_0} & \cdots & \overline{S_tS_0} \\ \overline{S_1S_0} & 1 & & \\ \vdots & & \ddots & \vdots \\ \overline{S_tS_0} & \cdots & & 1 \end{bmatrix} . \tag{14}$$

(c) Now perform the decomposition $\hat{\Phi}_t=\hat{A}_t\hat{A}_t^T$, where $\hat{A}_t$ is a triangular matrix.

(d) Draw the components of the vector $\mathbf{N}_t=(n_0,n_1,\dots,n_t)^T$ independently from a normal distribution. Transform this vector according to

$$\boldsymbol{\phi}_t=\hat{A}_t\mathbf{N}_t \ , \tag{15}$$

which gives the right correlations for the components of $\boldsymbol{\phi}_t$.

(e) Compute the sample averages $\overline{\phi_\tau S_t}$ for $\tau=0,\dots,t$.

(f) Obtain the coefficients $K_{t0},\dots,K_{tt-1}$ by solving the system of linear equations:

$$\mathbf{x}_t=\mathbf{K}_t\hat{\Phi}_t \ , \tag{16}$$

with

$$\mathbf{x}_t=(\overline{\phi_0S_t},\dots,\overline{\phi_tS_t})^T$$

and

$$\mathbf{K}_t=(K(t,0),K(t,1),\dots,K(t,t-1),0)^T \ . \tag{17}$$

(g) Determine the internal fields from

$$h^k(t)=\phi^k(t)+\eta\sum_{\tau<t}K(t,\tau)S^k(\tau) \ . \tag{18}$$

Reiterate steps 2(a)–2(g).

To keep the numerical errors of the Monte Carlo integrations small, a sufficiently large number N_T of spin trajectories must be used. Since there are no interactions between these N_T spins, we can use $N_T=10^6$ without problems. Note, that for a finite SK model with N spins about N^2 couplings must be stored, which restricted simulations to about $N=10^5$ spins [18].

Let us assume for the moment that for times $t', \tau \leq t$ all $C(t',\tau)$ and $K(t',\tau)$ were known without errors. Then, since for time $t+1$, N_T additional, independent trajectories are generated (avoiding the storage of the spins), the additive quantities $C(t+1,\tau)$ would fluctuate around their true values with errors that vanish like $\sim N_T^{-1/2}$ as N_T grows large. A small uncertainty in the parameters at previous time steps may lead to additional, even systematic deviations, because, e.g., the Gaussian random variables will not be drawn from their correct distribution. However, we expect that this error propagation effect can be safely neglected over the temporal range ($t_f=100$) that was investigated in our simulations, when a large number of $N_T=10^6$ trajectories is used. Larger fluctuations can occur for the memory kernel $K(t,s)$ (see Fig. 6) at large t and $\eta \approx 1$. This happens when the dynamics becomes very slow and the linear system (16) is almost singular. However, being filtered by the summation in Eq. (18), these fluctuations have only a small effect on the magnetization. To support this assumption, ten independent runs of the algorithm for $\eta=1$ have been performed. The fluctuations of the magnetization $m(t)$ did not increase with time t; e.g., $\Delta m(2)=0.98\times 10^{-3}$ and $\Delta m(100)=0.96\times 10^{-3} \approx N_T^{-1/2}$. The error of the extrapolated remanent magnetization m_∞ obtained by curve fitting is also $\approx 10^{-3}$. A comparison of simulations with different N_T will be given in a forthcoming paper (which treats the problem of nonzero temperature).

We will end this section with a brief discussion of the applicability of the present Monte Carlo method and its limitations.

Obviously, our method can be applied only to systems for which mean-field theory holds. In such cases an exact single-site dynamics can be formulated. This is true whenever there are no length scales in the problem. This includes mainly infinite range (fully connected) models, but also works for a finite coordination number, when the connectivity is completely random [30]. For short range models a mean-field description can only be an approximation.

At present, the method is formulated for parallel dynamics only, which for computational purpose is the most convenient choice. Models which are described by differential equations in time, like the soft-spin version of the SK model [24], will be harder to simulate. Single-site equations have to be discretized in time, which may lead to large dimensions for matrices K and C. However, when the dynamics becomes slow, larger time steps may be used. Finally, let us mention Glauber dynamics, which in physical models is used to describe the dynamics of Ising spins. A mean-field theory for the corresponding SK model has been formulated in [26], where the single-site problem involves averages over a complex Gaussian measure. At present, it is unclear whether such an average can be simulated efficiently by a stochastic process.

V. RESULTS AND DISCUSSION

A. Magnetization

Using our Monte Carlo procedure we have estimated the decay of the magnetization $m(t)$ for the first 100 time steps (Fig. 1). Since $m(t)=0$ for odd times, we consider $m(t)$ only for even t. The results of our simulations can be fitted well [Figs. 2(a) and 2(b)] by the function

$$m(t)=m_\infty+\mathrm{const}\times t^{-a}\exp(-t/\tau) , \tag{19}$$

where the parameters a, τ, and m_∞ are functions of the symmetry η. m_∞ is the extrapolated value of the magnetization for $t\to\infty$, i.e., the remanent magnetization. The values of m_∞ as a function of η are shown in Fig. 3. We find a clear transition from positive values of m_∞ for $\eta>\eta_c=0.825$ to $m_\infty=0$ for η below this value. This confirms the result of Spitzner and Kinzel [16] and of Pfenning, Rieger, and Schreckenberg [17], which were obtained from a finite size scaling of the remanent magnetization for models with finite N.

The picture of a nonequilibrium transition at η_c is further supported by an analysis of the characteristic time τ (Fig. 4). For $\eta<\eta_c$ the magnetization decays rapidly to zero with a finite relaxation time τ. If η is increased, we find a divergence of τ as η approaches η_c from below.

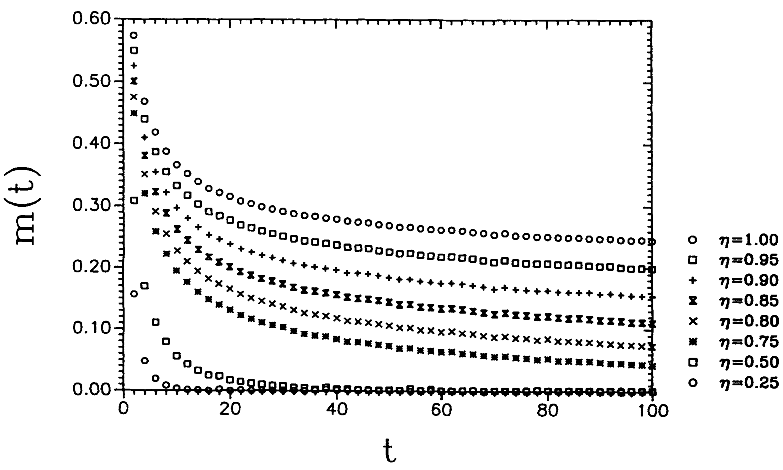

FIG. 1. Magnetization at even times for different values of η.

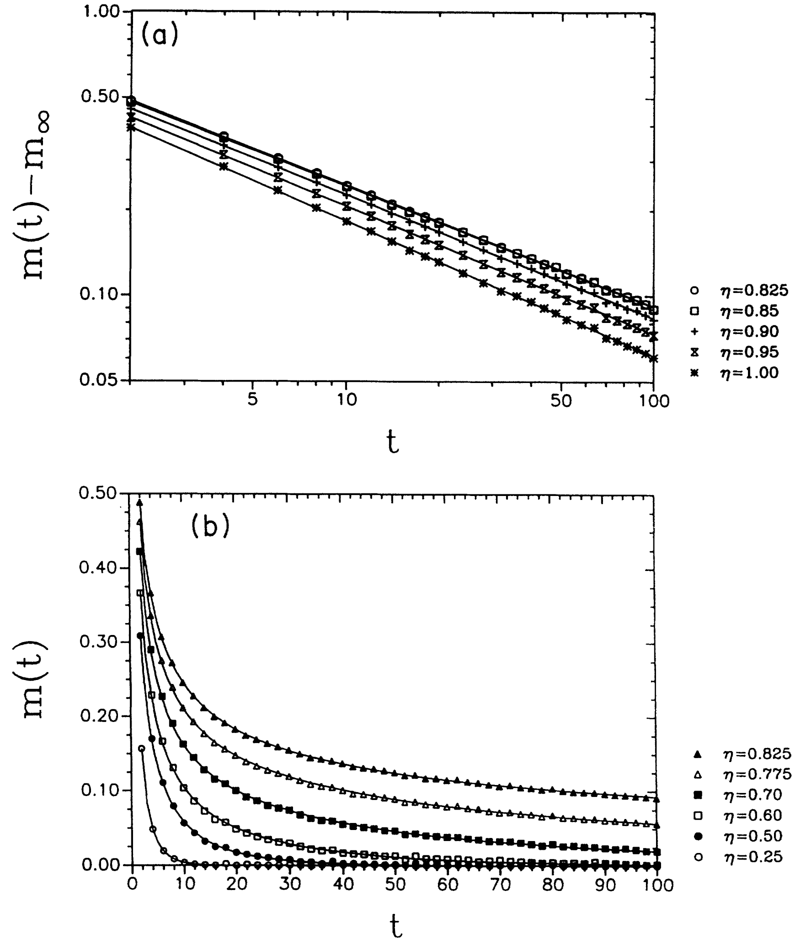

FIG. 2. (a) A log-log plot of the magnetization at even t for $\eta \geq \eta_c = 0.825$, together with the fit [Eq. (19)] (solid lines). (b) Magnetization at even t for $\eta \leq \eta_c = 0.825$, together with the fit [Eq. (19)] (solid lines).

For all $\eta > \eta_c$ the decay of the magnetization is a pure power law with an η-dependent exponent a (see Fig. 5):

$$m(t) = m_\infty + \text{const} \times t^{-a} . \tag{20}$$

The strong memory effect to the initial conditions at η close to 1 is directly reflected in the behavior of the memory kernel $K(t,s)$. As can be seen from Appendix A, this function describes the average response of the magnetization at time t to small variations of an external field at time s. For $\eta \ll 1$, memory effects are short ranged and K decays after a few steps. We have displayed $K(100,t)$ for $\eta = 1$ as a function of t in Fig. 6. Apart

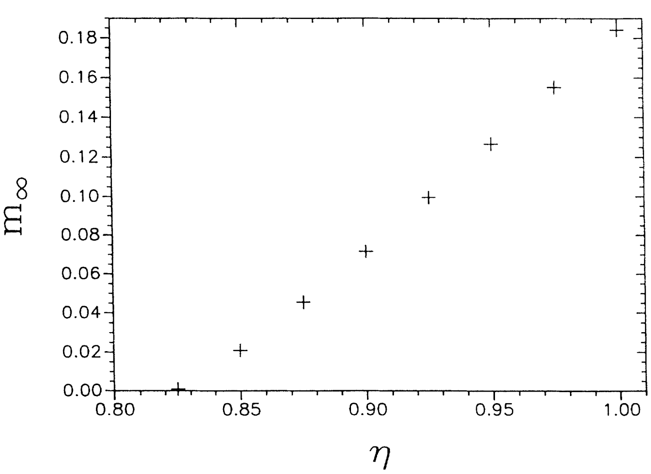

FIG. 3. Remanent magnetization as a function of the symmetry.

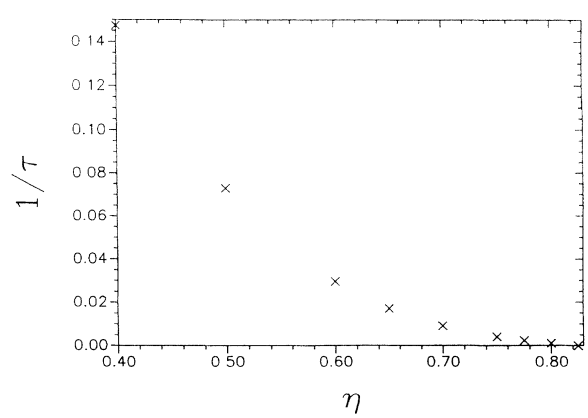

FIG. 4. Inverse of the relaxation time τ [See Eq. (19)] for the magnetization.

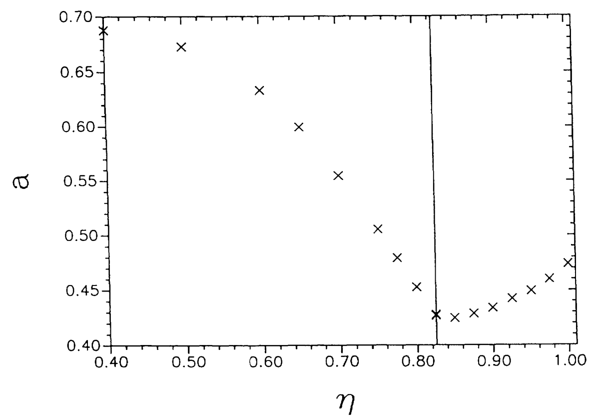

FIG. 5. Exponent for the power law of the magnetization, Eq. (19).

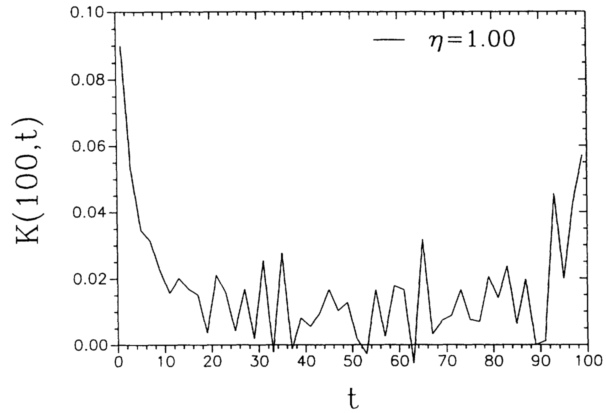

FIG. 6. Memory kernel $K(100,t)$ for $\eta=1$. We have connected the values at odd times t by straight lines. The values $K(100,t)=0$ at even t are not displayed.

from the fluctuations in the middle of the plot, which are due to numerical noise, a drastic increase of K at small t can be observed, showing a stronger memory of the initial conditions.

B. Dynamics at long times

We next try to find what type of attractor is approached by the system for long times. For $\eta=1$, we know that the dynamics relaxes into a cycle of length 2. Motivated by this exact result, we have studied the correlation function

$$C_2(t) \equiv N^{-1} \sum_{i=1}^{N} S_i(t-1) S_i(t+1)$$
$$= \langle S(t-1) S(t+1) \rangle ,$$

which converges to the value 1 for $t \to \infty$ if the system ends in a 2-cycle. Again the results of our simulations for this function could be very well fitted (Fig. 7) by the simple relation

$$C_2(t) = C_2(\infty) + \text{const} \times t^{-\gamma} . \tag{21}$$

In contrast to the magnetization, the decay of C_2 is described by an (η-dependent) power law (see also [17]) over the entire region of η (although the constant becomes small for small η).

In Fig. 8 we have displayed $C_2(\infty)$ as a function of η. It reaches the value 1 only for $\eta=1$. The curve is in excellent agreement with the simulations presented in [13]. Even in the nonergodic region $\eta_c < \eta < 1$ the system does *not completely freeze* into 2-cycles; some randomness in the spin flips is still persistent.

In order to gain more insight into the nature of the freezing transition at η_c, we have analyzed the distribution of spin-flip frequencies during the observation time of $t_f = 100$ time steps. This distribution was sampled from 10^5 independent realizations of the single spin dynamics (8). Following the arguments of Sec. III this is equivalent to sampling the spin flips at 10^5 *different* sites from *a single* infinite SK model. As the result, in Fig. 9 we show the distribution $P(n)$, which gives the fraction of spins which have flipped exactly n times during the 100 time steps of our simulation. [The fact that $P(n)$ is

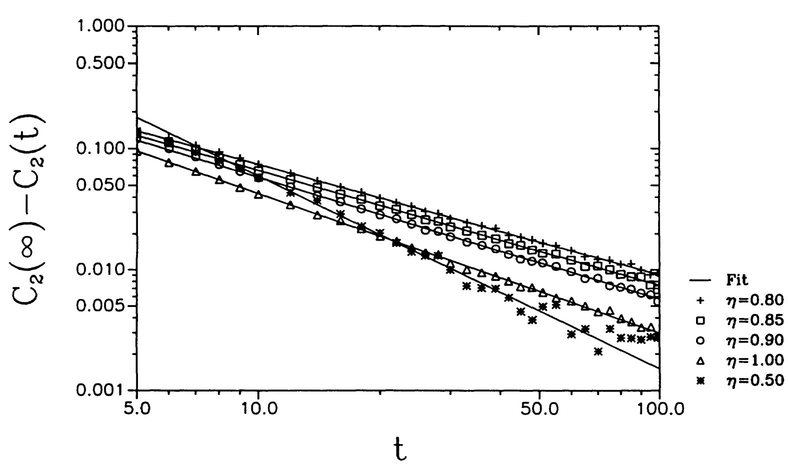

FIG. 7. A log-log plot of the two-step correlation function. The lines represent the fit [Eq. (21)].

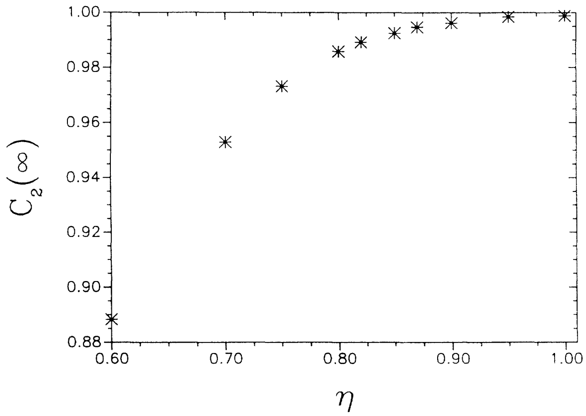

FIG. 8. Extrapolated limiting values of the two-step correlation function $C_2(t)$ [Eq. (21)] for infinite times.

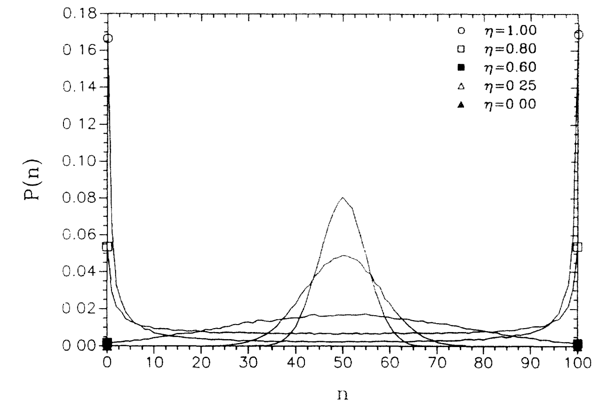

FIG. 9. Distribution of spin-flip frequencies during the first 100 steps.

symmetric around the point $n = t_f/2$ can be understood from the symmetry properties of the single spin dynamics; see Appendix C.]

The most striking features of the curves are the two peaks located at $n = 0$ and 100, which occur for large values of η and correspond to a finite fraction of spins which almost never, or almost always, perform flips. The dynamics of such spins represents cycles of length 2. Such spins clearly contribute to the remanent magnetization, because after an even number of steps they assume their initial states. Clearly for $\eta = 1$ all spin trajectories belong to these peaks at large times. If η is lowered, the weight of the two peaks becomes smaller, and an additional broad distribution of spin flip frequencies develops. Such trajectories represent more or less random sequences of spin flips. For $\eta \simeq 0.6$ the "2-cycle" peaks have disappeared.

Due to the limited simulation time we cannot precisely locate the value of η for which the peaks vanish for the first time at $t = \infty$. But it seems reasonable that this will be again *the critical value* η_c. Below this value cycles of length 2 have a vanishing probability. For the extreme case $\eta = 0$, the spin flips are completely randomized and $P(n)$ becomes a binomial distribution.

These simulations suggest the following picture: The introduction of asymmetry at zero temperature will leave a finite fraction of spins in an ordered state. This ordered "cluster" of spins melts down and vanishes when η is decreased to its critical value. Below η_c, the SK model with asymmetric couplings is fully ergodic. The dynamics of a *single spin becomes a stationary process* for long times. This can be also seen from Fig. 10, where the correlations $C(t, t-\Delta t) = \langle S(t) S(t-\Delta t) \rangle$ are displayed as functions of the time lag Δt for three different times t. The curves show the convergence to a steady-state correlation function depending on time differences only. Whether the transition at η_c leads to a divergence of the corresponding correlation time will require longer simulation times and cannot be judged from our present data. Alternatively we may use the ansatz of stationarity for the mean-field equations to obtain self-consistent solutions for correlation functions and memory kernels in the ergodic region. Such an approach will be discussed elsewhere.

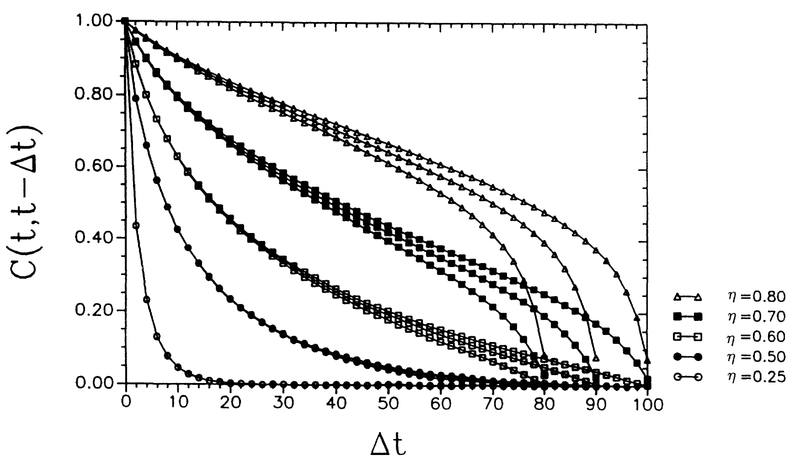

FIG. 10. Correlation function $C(t, t-\Delta t)$ for $t = 80$, 90, and 100.

VI. SUMMARY AND OUTLOOK

We have applied an alternative method to the infinite ($N=\infty$) SK model with asymmetric interactions. Using Monte Carlo simulations of exact dynamical mean-field equations, we have avoided the finite size effects of the simulations for large disordered spin systems. In addition, there is no need to store or recompute a huge number ($\approx N^2$) of quenched random interactions.

We have shown that the system undergoes a freezing transition as a function of asymmetry. The critical symmetry parameter $\eta_c=0.825$ separates a frozen phase with finite remanent magnetization from an ergodic phase where the system loses its memory from the initial state. In the frozen phase a cluster of spins in ordered 2-cycles coexists with spins that perform random flips with finite correlation times. Below η_c, the ordered cluster has completely vanished.

Further investigations will include the effects of nonzero temperatures which can easily be incorporated in the formalism. To explore the geometry of the phase space in more detail, we can also simulate the time evolution of a number of (real) replicas of the system.

A further interesting application will be the problem of learning in neural networks. It is not clear in general how the static results derived from replica theory are relevant for the dynamics [31–33] of learning algorithms when replica symmetry is broken (see, e.g., [34]).

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APPENDIX A: DERIVATION OF DYNAMICAL MEAN-FIELD EQUATIONS

To perform the average over the J_{ij}'s, one uses the integral representation for the δ functions and introduces auxiliary fields $\hat{h}_i(t)$ conjugate to the internal fields $h_i(t)$:

$$[Z(1)]_J \propto \left[\mathrm{Tr}_{S(t)} \int \prod_{i,t} \left\{ dh_i(t) d\hat{h}_i(t) \Theta(S_i(t+1)h_i(t)) \right. \right. \\ \left. \left. \times \exp\left[i\hat{h}_i(t) \left[h_i(t) - \sum_{j\neq i} \left\{ \left[\frac{1+\eta}{2} \right]^{1/2} J_{ij}^s + \left[\frac{1-\eta}{2} \right]^{1/2} J_{ij}^a \right\} S_j(t) \right] \right] \right\} \exp\left[i \sum_{i,t} l_i(t) h_i(t) \right] \right]_J \tag{A1}$$

where in the last exponential only the fields $l_i(t)$ at the sites $i=1,\ldots,N_T$ are different from zero. The average results in

$$[Z(1)]_J \propto \mathrm{Tr}_{S(t)} \int \prod_{i,t} \left\{ dh_i(t) d\hat{h}_i(t) \Theta(S_i(t+1)h_i(t)) \right\} \exp\left[i \sum_{i,t} (l_i(t)h_i(t) + \hat{h}_i(t)h_i(t)) \right] \\ \times \exp\left[-\frac{1}{2N} \sum_{i,j\neq i} \sum_{s,t} \hat{h}_i(t)\hat{h}_i(s) S_j(t) S_j(s) - \frac{\eta}{2N} \sum_{i,j\neq i} \sum_{s,t} \hat{h}_i(t) S_i(s) \hat{h}_j(s) S_j(t) \right] . \tag{A2}$$

Introducing order parameters $C(t,s)$ and $K(t,s)$

$$C(t,s) = \frac{1}{N} \sum_j S_j(t) S_j(s) , \\ K(t,s) = -\frac{i}{N} \sum_j \hat{h}_j(s) S_j(t) , \tag{A3}$$

together with their conjugates $\hat{C}(t,s)$ and $\hat{K}(t,s)$, and neglecting terms of order N^{-1}, we obtain

$$[Z(1)]_J \propto \int \prod_{t,s} (N dC(t,s) d\hat{C}(t,s) iN dK(t,s) d\hat{K}(t,s)) \\ \times \exp\left\{ iN \sum_{t,s} [C(t,s)\hat{C}(t,s) + iK(t,s)\hat{K}(t,s)] + \sum_i \ln[\tilde{Z}_i(l_i; C, \hat{C}, K, \hat{K})] \right\} , \tag{A4}$$

where the single-site partition function $\tilde{Z}_i$ is given by

$$\tilde{Z}_i(l_i;C,\hat{C},K,\hat{K}) \propto \mathrm{Tr}_{S_i(t)} \int \prod_t \left\{ dh_i(t)d\hat{h}_i(t)\Theta(S_i(t+1)h_i(t)) \right\} \exp[i \sum_t (l_i(t)h_i(t)+\hat{h}_i(t)h_i(t))]$$
$$\times \exp\left[-\tfrac{1}{2}\sum_{s,t} C(t,s)\hat{h}_i(t)\hat{h}_i(s) - i\sum_{s,t}\hat{C}(t,s)S_i(t)S_i(s) - \frac{i\eta}{2}\sum_{s,t} K(t,s)\hat{h}_i(t)S_i(s) - i\sum_{s,t}\hat{K}(t,s)\hat{h}_i(s)S_i(t) \right] . \quad \text{(A5)}$$

At this point the dynamical variables are decoupled with respect to their site index i.

In the limit $N\to\infty$ the integrations over $C(t,s)$, $\hat{C}(t,s)$, $K(t,s)$, and $\hat{K}(t,s)$ can be performed with the saddle-point method. The stationary values of the order parameters are found from the following set of equations:

$$\hat{C}(t,s) = -\frac{i}{2N}\sum_i \langle \hat{h}_i(t)\hat{h}_i(s)\rangle_{\tilde{Z}_i} , \quad \text{(A6)}$$

$$C(t,s) = \frac{1}{N}\sum_i \langle S_i(t)S_i(s)\rangle_{\tilde{Z}_i} , \quad \text{(A7)}$$

$$\hat{K}(t,s) = -\frac{i\eta}{2N}\sum_i \langle \hat{h}_i(t)S_i(s)\rangle_{\tilde{Z}_i} , \quad \text{(A8)}$$

$$K(t,s) = -\frac{i}{N}\sum_i \langle \hat{h}_i(s)S_i(t)\rangle_{\tilde{Z}_i} . \quad \text{(A9)}$$

$\langle\ \rangle_{\tilde{Z}_i}$ denotes an average with respect to the single-site partition function.

It can be shown that the first saddle-point equation has only the trivial solution $\hat{C}(t,s)=0$, any other solution would violate the normalization $[Z(\mathbf{l}=\mathbf{0})]=1$.

From (A8) and (A9) we find

$$\hat{K}(t,s) = \frac{\eta}{2}K(s,t) . \quad \text{(A10)}$$

Omitting the single-site partition functions with $l_i=0$ (collecting all prefactors it can be shown that they are equal to 1), we arrive at

$$[Z(\mathbf{l})]_J \propto \prod_{i=1}^{N_T} \mathrm{Tr}_{S_i(t)} \int \prod_t \left\{ dh_i(t)d\hat{h}_i(t)\Theta(S_i(t+1)h_i(t)) \right\} \exp\left[\sum_t (l_i(t)h_i(t)+\hat{h}_i(t)h_i(t)) \right]$$
$$\times \exp\left[-\tfrac{1}{2}\sum_{s,t} C_{ts}\hat{h}_i(t)\hat{h}_i(s) - i\eta\sum_{s,t} K_{ts}\hat{h}_i(t)S_i(s) \right] . \quad \text{(A11)}$$

The generating function (A11) describes a system of N_T *noninteracting* spins. It can be rewritten in a form where each spin is coupled to an effective stochastic field.

We linearize the quadratic terms in $\hat{h}_i(t)$ by introducing Gaussian random variables $\phi_i(t)$, independently for each i with zero mean and covariance $\langle \phi_i(t)\phi_i(s)\rangle_\phi = C(t,s)$. Using the identity

$$\exp\left\{ -\tfrac{1}{2}\sum_{s,t} \langle \phi_i(t)\phi_i(s)\rangle_\phi \hat{h}_i(t)\hat{h}_i(s) \right\} = \left\langle \exp\left\{ -i\sum_t \phi_i(t)\hat{h}_i(t) \right\} \right\rangle_\phi , \quad \text{(A12)}$$

where $\langle\ \rangle_\phi$ denotes an average over the time-dependent Gaussian random variables $\phi_i(t)$, and integrating over the auxiliary fields $\hat{h}_i(t)$, we arrive at Eq. (7).

Rewriting the order parameter equations (A7) and (A9) as Gaussian averages, using the fact that all [those with $l_i(t)=0$] but a finite number of single-site averages give the same value, the correlation function reads

$$C(t,s) = \langle \phi(t)\phi(s)\rangle_\phi = \langle S(t)S(s)\rangle_\phi . \quad \text{(A13)}$$

Similarly, the kernel $K(t,s)$ can be expressed by an average over the fields $\phi_i(t)$. Comparing (A12) and (A11), we see that each term $-i\hat{h}(s)$ can be replaced by a derivative $\partial/\partial\phi(s)$. Thus

$$K(t,s) = -i\langle \hat{h}(s)S(t)\rangle_\phi = \left\langle \frac{\partial}{\partial\phi(s)}S(t) \right\rangle_\phi . \quad \text{(A14)}$$

This relation also shows that $K(t,s)$ is the average response of the magnetization at time t with respect to a small variation of an external field at time s.

APPENDIX B: A THEOREM FOR GAUSSIAN RANDOM VARIABLES

Let $\boldsymbol{\phi}$ be a vector with r components ϕ_i which are Gaussian distributed with zero mean $\langle \phi_i \rangle = 0$ and covariance matrix $\langle x_k x_j \rangle = C_{kj}$. We show for functions f

of these random variables that

$$\langle \phi_k f(\boldsymbol{\phi}) \rangle = \sum_j C_{kj} \left\langle \frac{\partial f(\boldsymbol{\phi})}{\partial \phi_j} \right\rangle . \tag{B1}$$

Integrating the right-hand side of Eq. (B1) by parts, we obtain

$$\begin{aligned} \sum_j C_{kj} \left\langle \frac{\partial f(\boldsymbol{\phi})}{\partial \phi_j} \right\rangle &\propto \sum_j C_{kj} \int \frac{\partial f(\boldsymbol{\phi})}{\partial \phi_j} \exp \left[-\tfrac{1}{2} \sum_{pq} C_{pq}^{-1} \phi_p \phi_q \right] d^r\phi \\ &\propto -\sum_j C_{kj} \int f(\boldsymbol{\phi}) \frac{\partial}{\partial \phi_j} \exp \left[-\tfrac{1}{2} \sum_{pq} C_{pq}^{-1} \phi_p \phi_q \right] d^r\phi \\ &\propto \sum_j C_{kj} \int f(\boldsymbol{\phi}) \sum_l (\phi_l C_{jl}^{-1}) \exp \left[-\tfrac{1}{2} \sum_{pq} C_{pq}^{-1} \phi_p \phi_q \right] d^r\phi , \end{aligned} \tag{B2}$$

where we have omitted the normalization constant for the Gaussian distribution. Performing the sum over j, we find

$$\begin{aligned} \sum_j C_{kj} \left\langle \frac{\partial f(\phi)}{\partial \phi_j} \right\rangle &\propto \int f(\boldsymbol{\phi}) \phi_k \exp \left[-\tfrac{1}{2} \sum_{pq} C_{pq}^{-1} \phi_p \phi_q \right] d^r\phi \\ &= \langle \phi_k f(\boldsymbol{\phi}) \rangle , \end{aligned} \tag{B3}$$

which is the desired result.

Applying this theorem to the function $f(\phi) = S(t) \equiv S(t;\boldsymbol{\phi})$ and using (10) yields Eq. (11).

APPENDIX C: SYMMETRY OF ORDER PARAMETERS

We prove the symmetry properties of $K(t,s)$ and $C(t,s)$ by induction:

It is easy to see that $\langle S(1)S(0) \rangle_\phi = 0$. We assume for $t = \text{even}$, i.e., $t+1 = \text{odd}$, that $\langle S(\text{odd})S(\text{even}) \rangle_\phi = 0$ for all odd , even $\leq t$, so that Gaussian variables at odd and even times are decoupled. We then perform the transformation $\phi(\text{odd}) \to -\phi(\text{odd})$ for all Gaussian variables at odd times. This leaves the joint Gaussian density and the odd-time spins invariant but transforms $S(\text{even})$ into $-S(\text{even})$. So

$$\begin{aligned} \langle S(t+1)S(\text{even}) \rangle &= \langle S(\text{odd})S(\text{even}) \rangle \\ &= -\langle S(\text{odd})S(\text{even}) \rangle = 0 . \end{aligned} \tag{C1}$$

To treat $K(t,s)$, we use that $\langle S(\text{odd})\phi(\text{odd}') \rangle = 0$. The vector $K(t+1,s)$, $s = 1,3,5, \ldots$ is a solution of (11):

$$\begin{aligned} 0 &= \langle S(t+1)\phi(\text{odd}') \rangle \\ &= \langle S(\text{odd})\phi(\text{odd}') \rangle \\ &= \sum_{\text{odd}''} K(\text{odd}, \text{odd}'')C(\text{odd}'', \text{odd}') . \end{aligned} \tag{C2}$$

Since the matrix C is positive definite, the linear system has the only solution $K(t+1, \text{odd}) = 0$, from which the relation is proved. The same arguments apply for $t = \text{odd}$.

Finally we prove a simple relation for the average number of spin flips during a time interval Δt from these symmetry properties.

For a given trajectory $\mathcal{T}_1$ of spins the corresponding trajectory $\mathcal{T}_2$, where all spins at odd-time steps are simultaneously reversed $[S(\text{odd}) \to -S(\text{odd})]$, must have the same probability as $\mathcal{T}_1$. Whenever a spin flip occurs in $\mathcal{T}_1$, it will be absent in $\mathcal{T}_2$, and vice versa. Since the total number of possible flips equals Δt, we obtain

$$\begin{aligned} &\Pr((\text{No. of flips}) = n) \\ &\quad = \Pr((\text{No. of flips}) = \Delta t - n) . \end{aligned} \tag{C3}$$

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